4.3 Minimum Spanning Trees

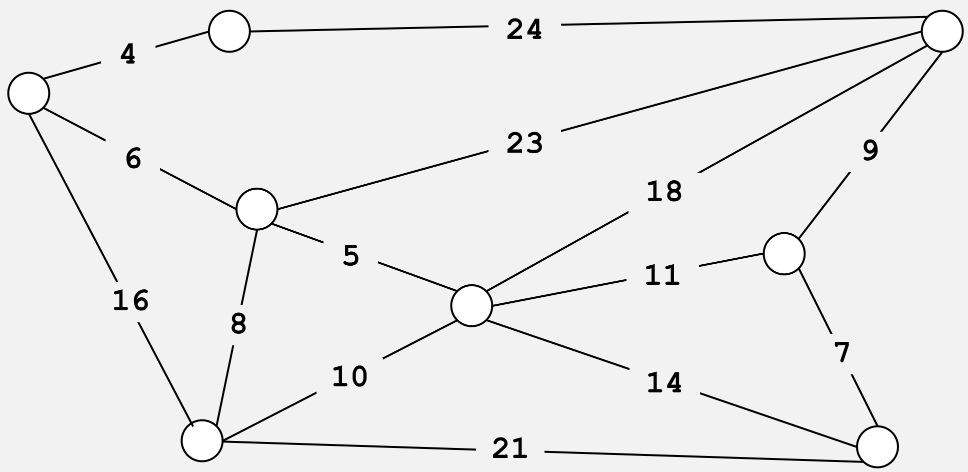
- edge-weighted graph API
- greedy algorithm
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

Minimum spanning tree

Given. Undirected graph G with positive edge weights (connected).

Def. A spanning tree of G is a subgraph T that is connected and acyclic.

Goal. Find a min weight spanning tree.



graph G

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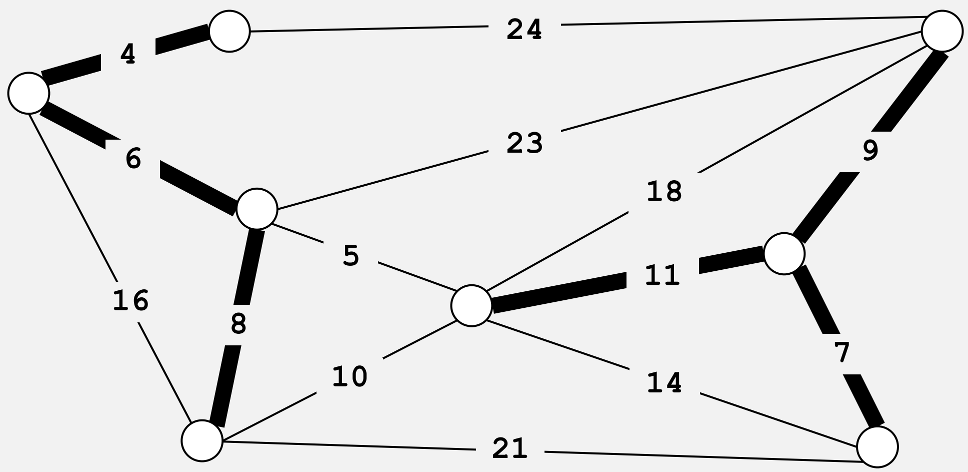

not connected

Minimum spanning tree

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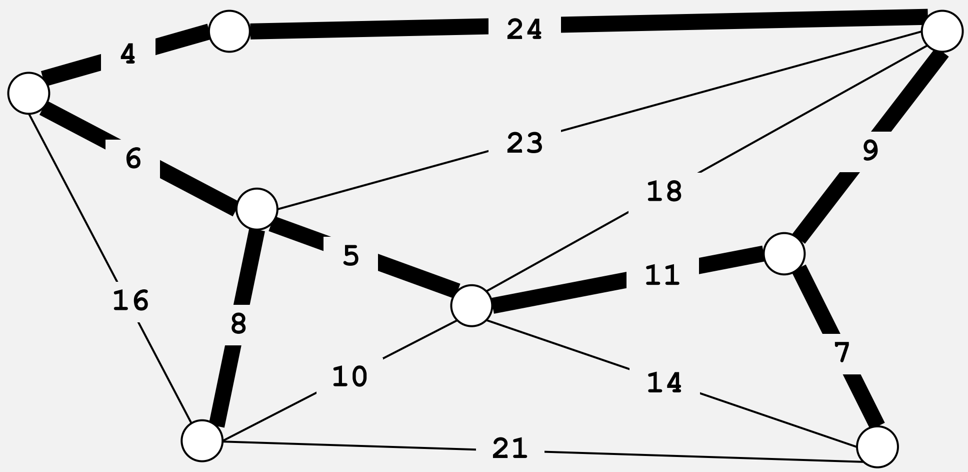
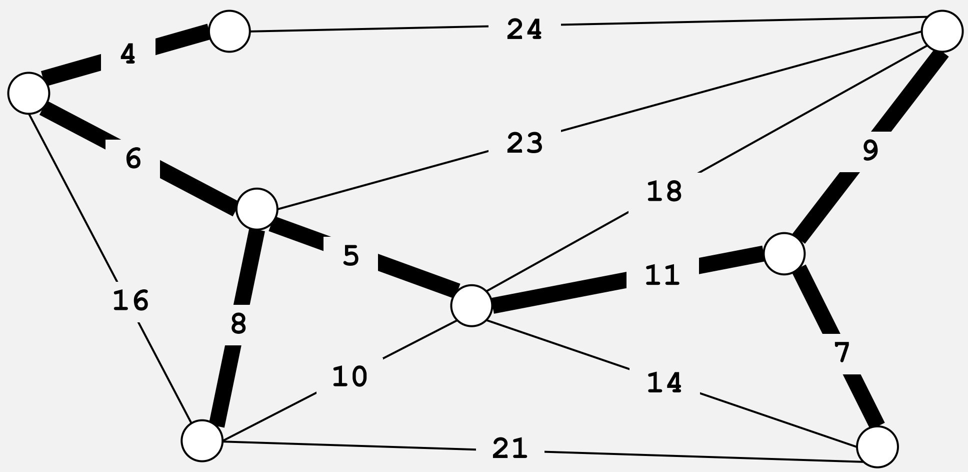


not acyclic

Given. Undirected graph G with positive edge weights (connected).

Def. A spanning tree of G is a subgraph T that is connected and acyclic.

Goal. Find a min weight spanning tree.

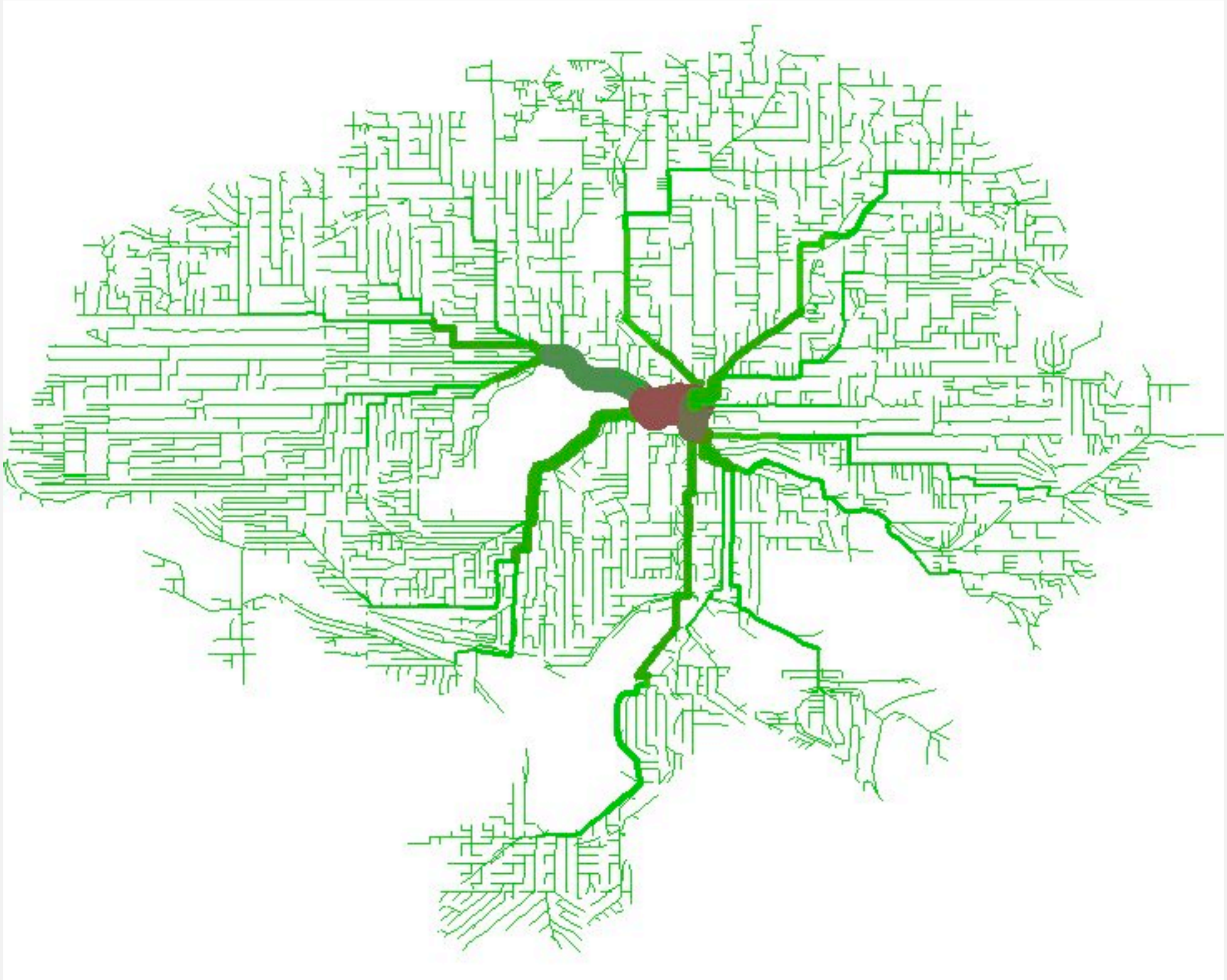


spanning tree T: cost = 50 = 4 + 6 + 8 + 5 + 11 + 9 + 7

Brute force. Try all spanning trees?

Network design

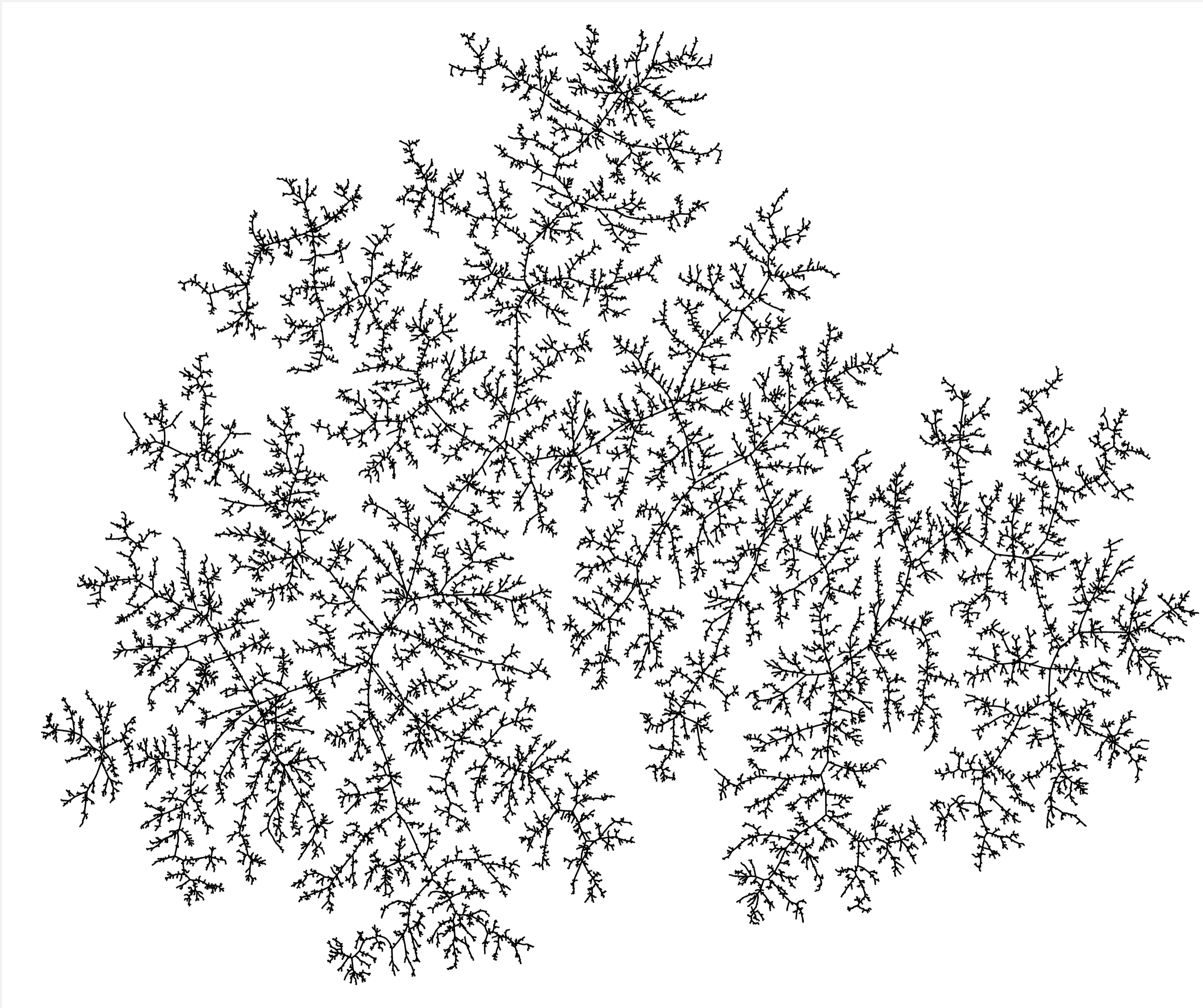
MST of bicycle routes in North Seattle



http://www.flickr.com/photos/ewedistrict/21980840

Models of nature

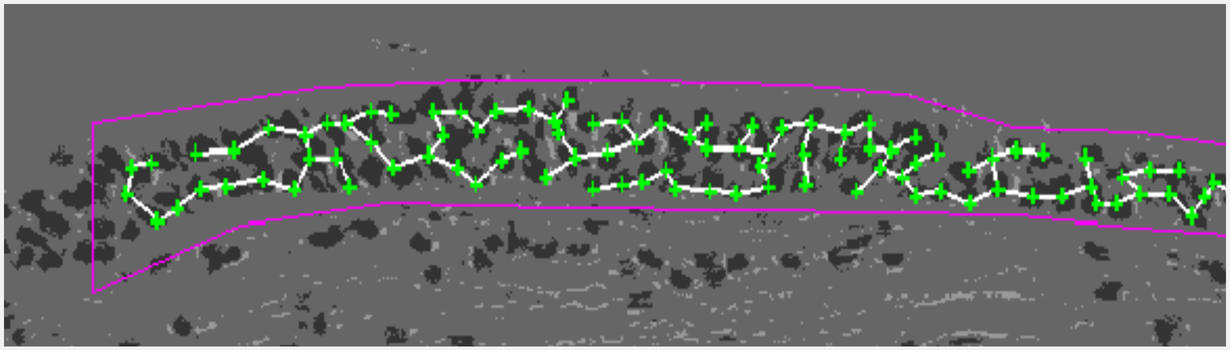
MST of random graph

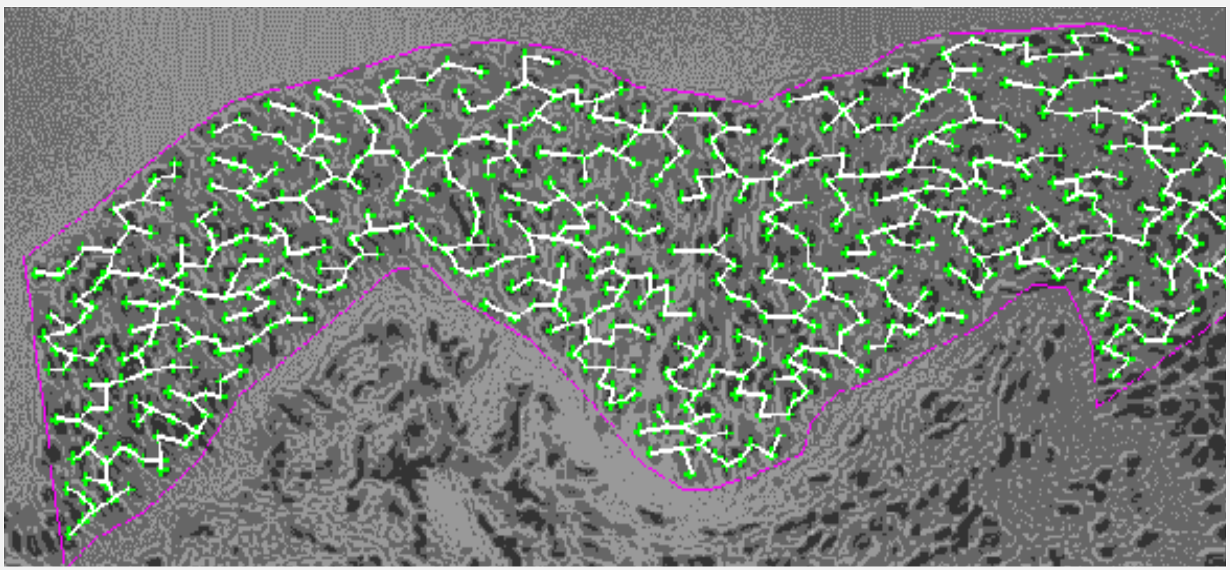


http://algo.inria.fr/broutin/gallery.html

Medical image processing

MST describes arrangement of nuclei in the epithelium for cancer research

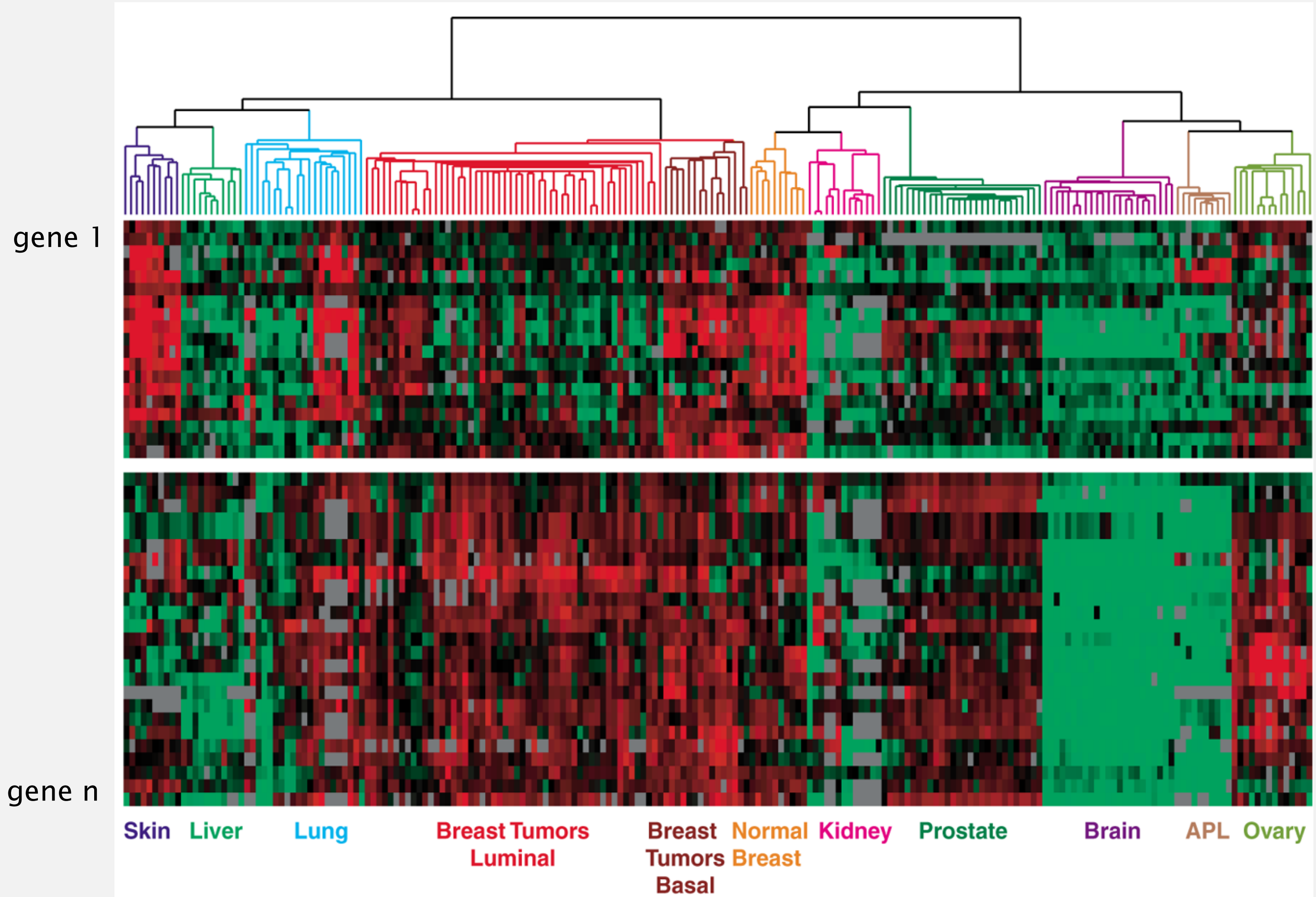




http://www.bccrc.ca/ci/ta01_archlevel.html

Dendrogram of cancers in human

Clustering of genes expressed in malignant tumors by tissue type



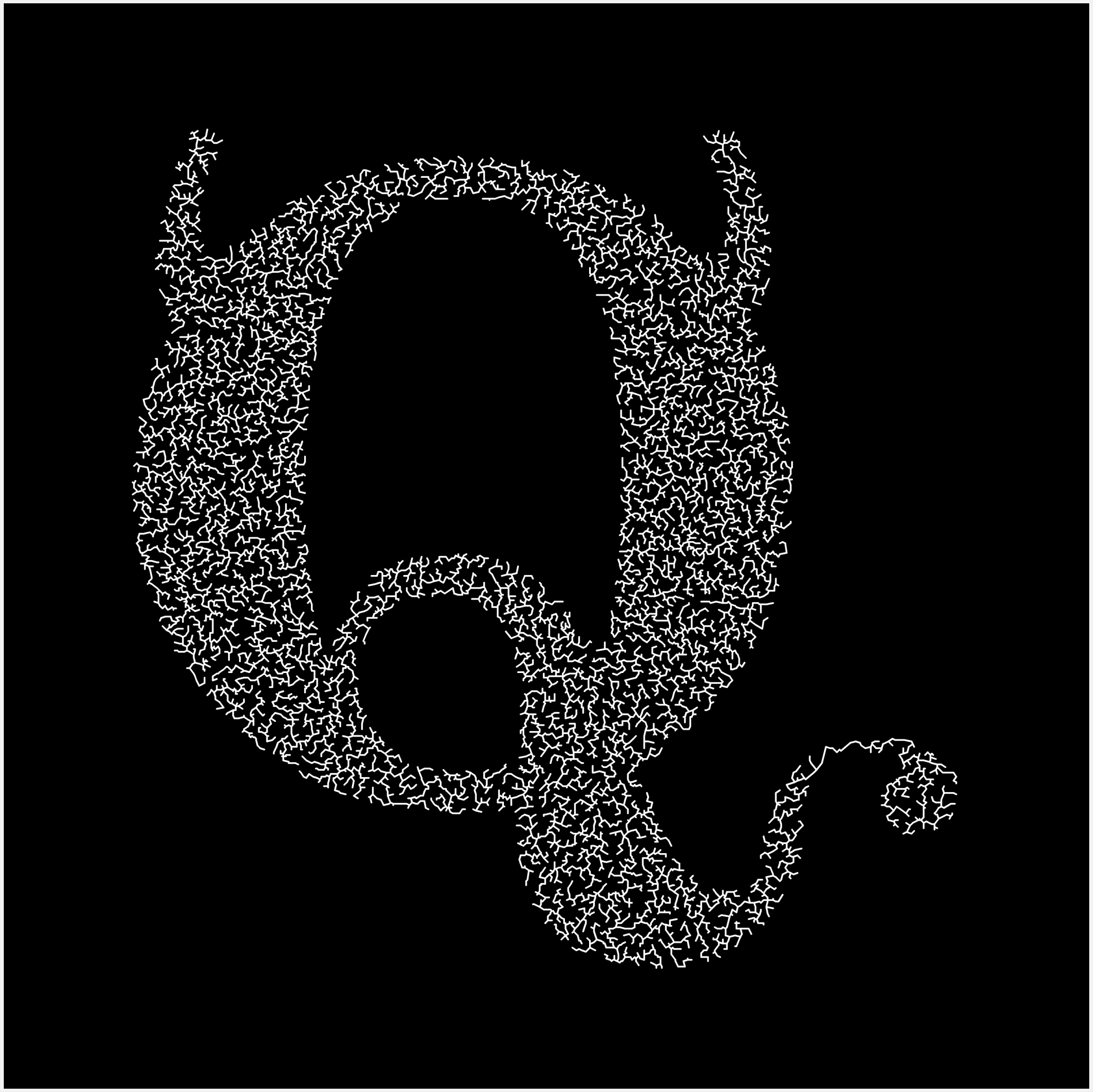
gene expressed

gene not expressed

Reference: Botstein & Brown group

Medical image processing

MST dithering



http://www.flickr.com/photos/quasimondo/2695389651

Applications

MST is fundamental problem with diverse applications.

- Dithering.
- Cluster analysis.
- Max bottleneck paths.
- Real-time face verification.
- LDPC codes for error correction.
- Image registration with Renyi entropy.
- Find road networks in satellite and aerial imagery.
- Reducing data storage in sequencing amino acids in a protein.
- Model locality of particle interactions in turbulent fluid flows.
- Autoconfig protocol for Ethernet bridging to avoid cycles in a network.
- Approximation algorithms for NP-hard problems (e.g., TSP, Steiner tree).
- Network design (communication, electrical, hydraulic, cable, computer, road).

http://www.ics.uci.edu/~eppstein/gina/mst.html

- **edge-weighted graph API**
- greedy algorithm
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

Weighted edge API

Edge abstraction needed for weighted edges.

```
public class Edge implements Comparable<Edge>
```

	`Edge(int v, int w, double weight)`	*create a weighted edge v-w*
`int`	`either()`	*either endpoint*
`int`	`other(int v)`	*the endpoint that's not v*
`int`	`compareTo(Edge that)`	*compare this edge to that edge*
`double`	`weight()`	*the weight*
`String`	`toString()`	*string representation*

Idiom for processing an edge `e`: `int v = e.either(), w = e.other(v);`

Weighted edge: Java implementation

```java
public class Edge implements Comparable<Edge>
{
   private final int v, w;
   private final double weight;

   public Edge(int v, int w, double weight)
   {
      this.v = v;
      this.w = w;
      this.weight = weight;
   }

   public int either()
   {  return v;  }

   public int other(int vertex)
   {
      if (vertex == v) return w;
      else return v;
   }

   public int compareTo(Edge that)
   {
      if      (this.weight < that.weight) return -1;
      else if (this.weight > that.weight) return +1;
      else                                return  0;
   }
}
```

constructor

either endpoint

other endpoint

compare edges by weight

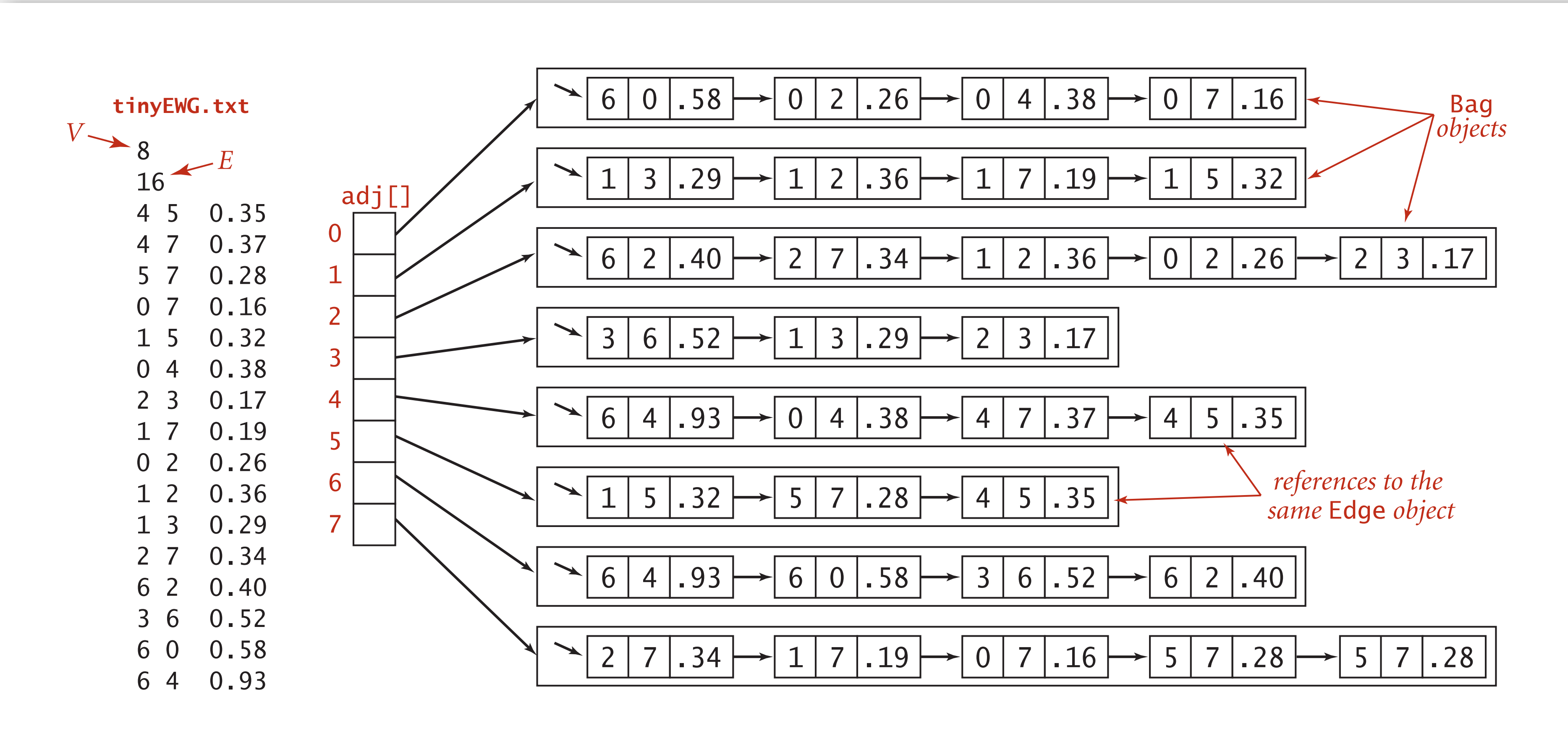
Edge-weighted graph API

	`public class EdgeWeightedGraph`	
	`EdgeWeightedGraph(int V)`	*create an empty graph with V vertices*
	`EdgeWeightedGraph(In in)`	*create a graph from input stream*
`void`	`addEdge(Edge e)`	*add weighted edge e to this graph*
`Iterable<Edge>`	`adj(int v)`	*edges incident to v*
`Iterable<Edge>`	`edges()`	*all edges in this graph*
`int`	`V()`	*number of vertices*
`int`	`E()`	*number of edges*
`String`	`toString()`	*string representation*

Conventions. Allow self-loops and parallel edges.

Edge-weighted graph: adjacency-lists representation

Maintain vertex-indexed array of **`Edge`** lists.



Edge-weighted graph: adjacency-lists implementation

```
public class EdgeWeightedGraph
{
   private final int V;
   private final Bag<Edge>[] adj;

   public EdgeWeightedGraph(int V)
   {
      this.V = V;
      adj = (Bag<Edge>[]) new Bag[V];
      for (int v = 0; v < V; v++)
         adj[v] = new Bag<Edge>();
   }

   public void addEdge(Edge e)
   {
      int v = e.either(), w = e.other(v);
      adj[v].add(e);
      adj[w].add(e);
   }

   public Iterable<Edge> adj(int v)
   {  return adj[v];  }
}
```

same as **`Graph`**, but adjacency lists of **`Edge`**s instead of integers

constructor

add edge to both adjacency lists

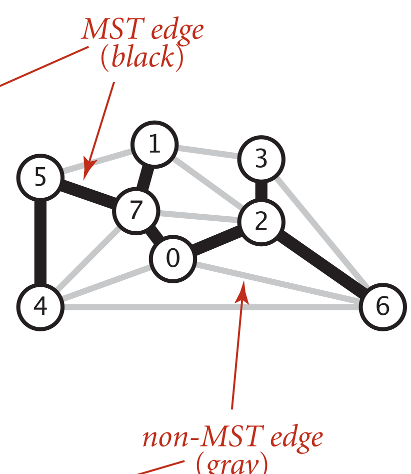
Minimum spanning tree API

Q. How to represent the MST?

`public class MST`		
	`MST(EdgeWeightedGraph G)`	*constructor*
`Iterable<Edge>`	`edges()`	*edges in MST*
`double`	`weight()`	*weight of MST*

tinyEWG.txt

```
8      <- V
16     <- E
4 5  0.35
4 7  0.37
5 7  0.28
0 7  0.16
1 5  0.32
0 4  0.38
2 3  0.17
1 7  0.19
0 2  0.26
1 2  0.36
1 3  0.29
2 7  0.34
6 2  0.40
3 6  0.52
6 0  0.58
6 4  0.93
```




```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```

Minimum spanning tree API

Q. How to represent the MST?

`public class MST`		
	`MST(EdgeWeightedGraph G)`	*constructor*
`Iterable<Edge>`	`edges()`	*edges in MST*
`double`	`weight()`	*weight of MST*

```
public static void main(String[] args)
{
   In in = new In(args[0]);
   EdgeWeightedGraph G = new EdgeWeightedGraph(in);
   MST mst = new MST(G);
   for (Edge e : mst.edges())
      StdOut.println(e);
   StdOut.printf("%.2f\n", mst.weight());
}
```

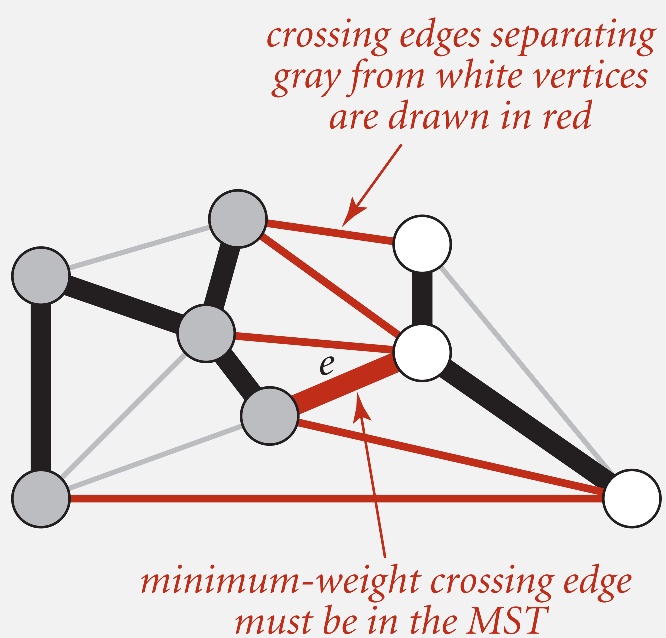
```
% java MST tinyEWG.txt
0-7 0.16
1-7 0.19
0-2 0.26
2-3 0.17
5-7 0.28
4-5 0.35
6-2 0.40
1.81
```

- edge-weighted graph API
- **greedy algorithm**
- Kruskal's algorithm
- Prim's algorithm
- advanced topics

Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets.
A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.



Cut property: correctness proof

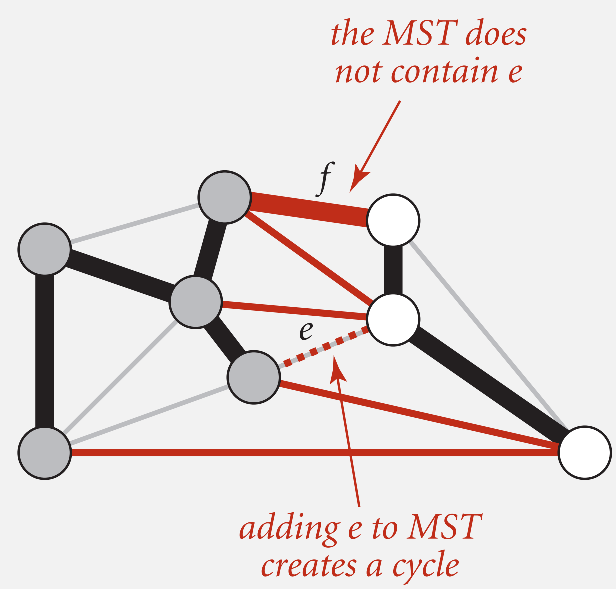
Simplifying assumptions. Edge weights are distinct; graph is connected.

Def. A cut in a graph is a partition of its vertices into two (nonempty) sets.
A crossing edge connects a vertex in one set with a vertex in the other.

Cut property. Given any cut, the crossing edge of min weight is in the MST.

Pf. Let e be the min-weight crossing edge in cut.

- Suppose e is not in the MST.
- Adding e to the MST creates a cycle.
- Some other edge f in cycle must be a crossing edge.
- Removing f and adding e is also a spanning tree.
- Since weight of e is less than the weight of f, that spanning tree is lower weight.
- Contradiction. ▪



Greedy MST algorithm demo

Greedy algorithm.

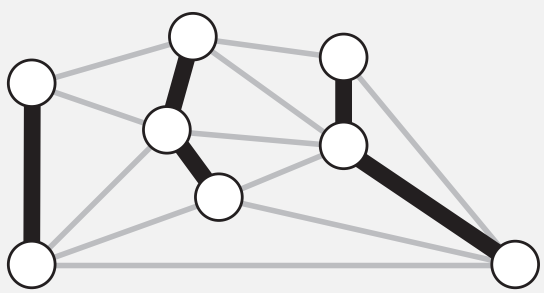
- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Continue until $V - 1$ edges are colored black.

Greedy MST algorithm: correctness proof

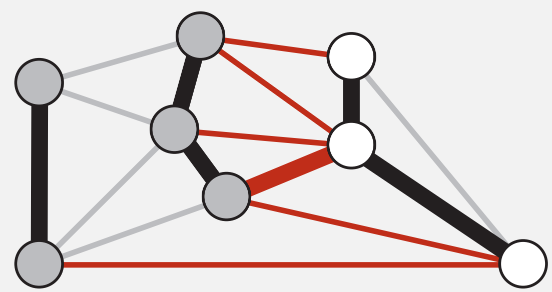
Proposition. The greedy algorithm computes the MST.

Pf.

- Any edge colored black is in the MST (via cut property).
- If fewer than $V - 1$ black edges, there exists a cut with no black crossing edges.
  (consider cut whose vertices are one connected component)



fewer than V-1 edges colored black



a cut with no black crossing edges

Greedy MST algorithm: efficient implementations

Proposition. The following algorithm computes the MST:

- Start with all edges colored gray.
- Find a cut with no black crossing edges, and color its min-weight edge black.
- Continue until $V - 1$ edges are colored black.

Efficient implementations. How to choose cut? How to find min-weight edge?

Ex 1. Kruskal's algorithm. [stay tuned]

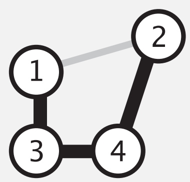
Ex 2. Prim's algorithm. [stay tuned]

Ex 3. Borüvka's algorithm.

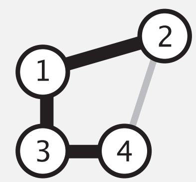
Q. What if edge weights are not all distinct?

A. Greedy MST algorithm still correct if equal weights are present!

(our correctness proof fails, but that can be fixed)



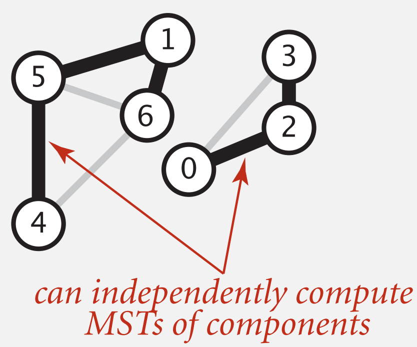
1	2	1.00
1	3	0.50
2	4	1.00
3	4	0.50



1	2	1.00
1	3	0.50
2	4	1.00
3	4	0.50

Q. What if graph is not connected?

A. Compute minimum spanning forest = MST of each component.



4	5	0.61
4	6	0.62
5	6	0.88
1	5	0.11
2	3	0.35
0	3	0.6
1	6	0.10
0	2	0.22

Greed is good

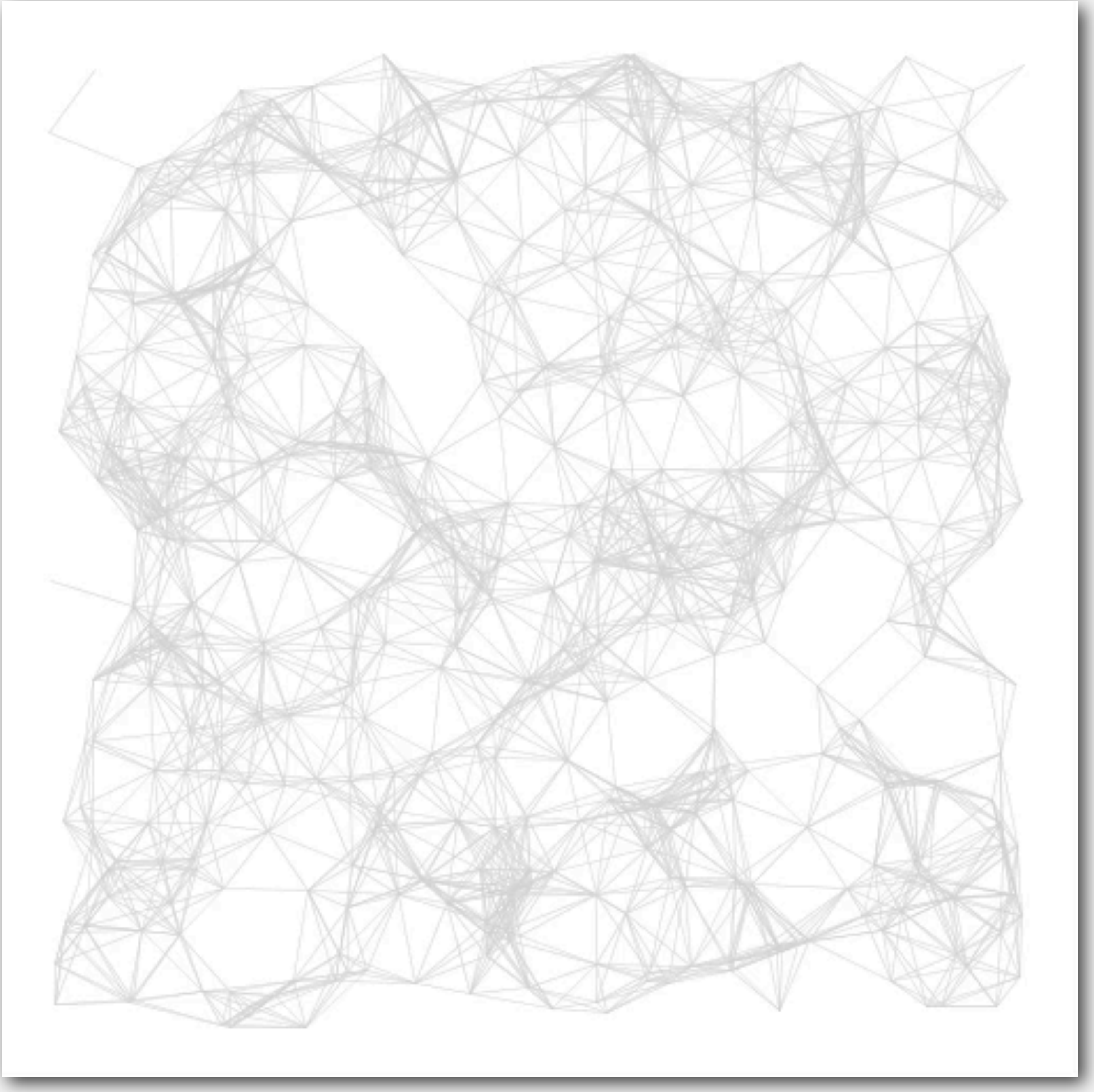
Gordon Gecko (Michael Douglas) address to Teldar Paper Stockholders in Wall Street (1986)

- edge-weighted graph API
- greedy algorithm
- **Kruskal's algorithm**
- Prim's algorithm
- advanced topics

Kruskal's algorithm. [Kruskal 1956]

- Consider edges in ascending order of weight.
- Add the next edge to the tree T unless doing so would create a cycle.

Kruskal's algorithm: visualization

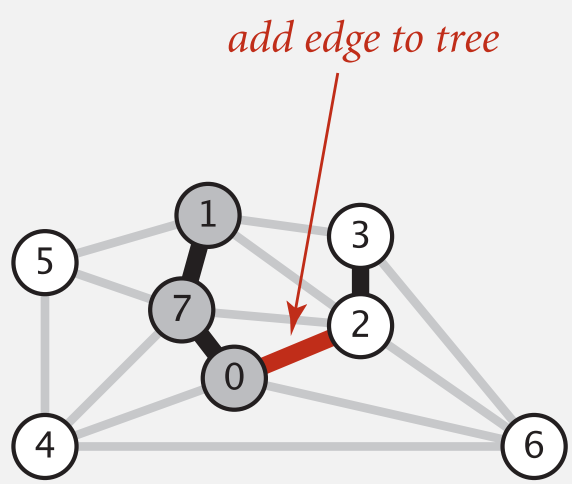


Kruskal's algorithm: correctness proof

Proposition. Kruskal's algorithm computes the MST.

Pf. Kruskal's algorithm is a special case of the greedy MST algorithm.

- Suppose Kruskal's algorithm colors the edge $e = v–w$ black.
- Cut = set of vertices connected to v in tree T.
- No crossing edge is black.
- No crossing edge has lower weight. Why?

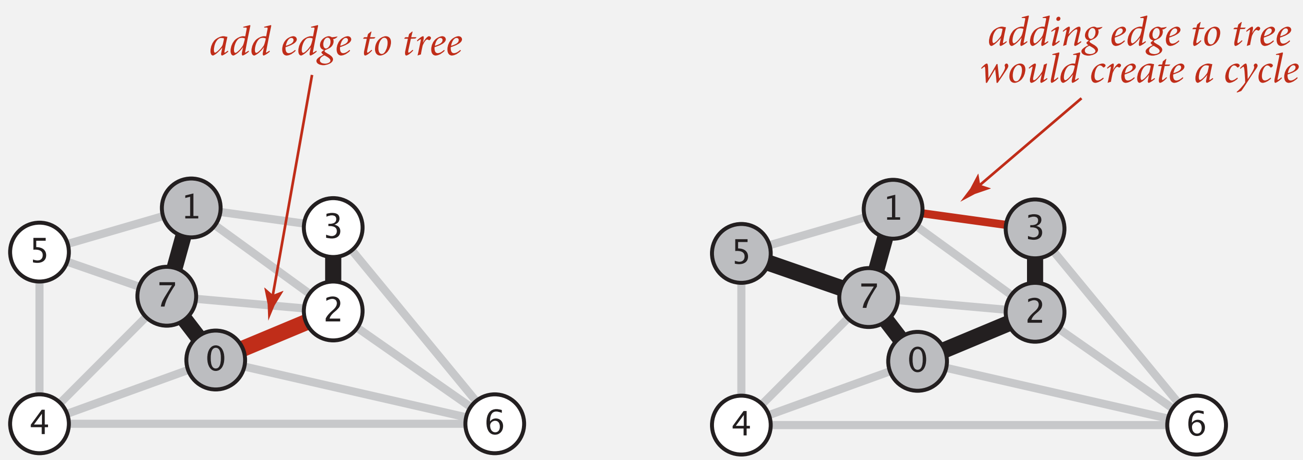


Kruskal's algorithm: implementation challenge

Challenge. Would adding edge $v–w$ to tree T create a cycle? If not, add it.

How difficult?

- $E + V$
- V ← run DFS from v, check if w is reachable (T has at most V – 1 edges)
- $\log V$
- $\log^* V$ ← use the union-find data structure !
- 1

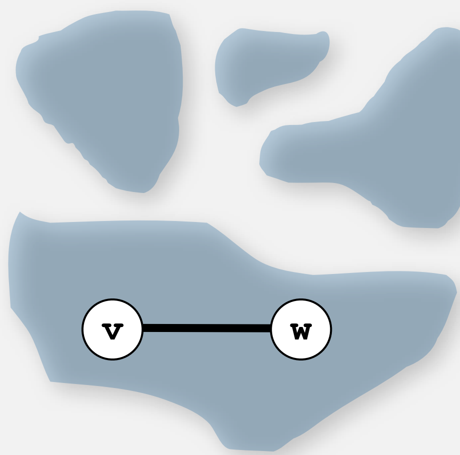


Kruskal's algorithm: implementation challenge

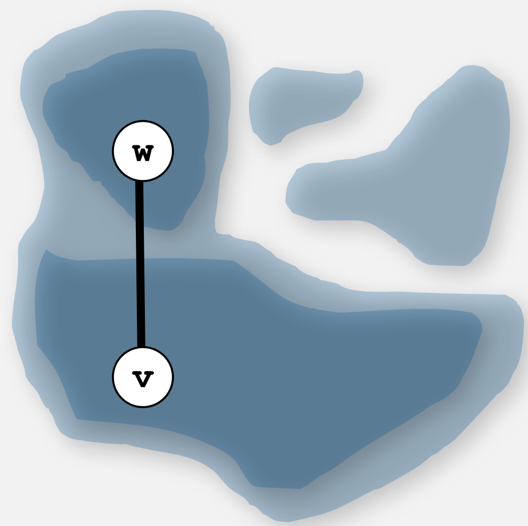
Challenge. Would adding edge $v–w$ to tree T create a cycle? If not, add it.

Efficient solution. Use the union-find data structure.

- Maintain a set for each connected component in T.
- If v and w are in same set, then adding $v–w$ would create a cycle.
- To add $v–w$ to T, merge sets containing v and w.



Case 1: adding v–w creates a cycle



Case 2: add v–w to T and merge sets containing v and w

Kruskal's algorithm: Java implementation

```
public class KruskalMST
{
   private Queue<Edge> mst = new Queue<Edge>();

   public KruskalMST(EdgeWeightedGraph G)
   {
      MinPQ<Edge> pq = new MinPQ<Edge>();
      for (Edge e : G.edges())
         pq.insert(e);

      UF uf = new UF(G.V());
      while (!pq.isEmpty() && mst.size() < G.V()-1)
      {
         Edge e = pq.delMin();
         int v = e.either(), w = e.other(v);
         if (!uf.connected(v, w))
         {
            uf.union(v, w);
            mst.enqueue(e);
         }
      }
   }

   public Iterable<Edge> edges()
   {  return mst;  }
}
```

build priority queue

greedily add edges to MST

edge v–w does not create cycle

merge sets

add edge to MST

Kruskal's algorithm: running time

Proposition. Kruskal's algorithm computes MST in time proportional to $E \log E$ (in the worst case).

Pf.

operation	frequency	time per op
build pq	1	E
delete-min	E	log E
union	V	log* V †
connected	E	log* V †

† amortized bound using weighted quick union with path compression

recall: log* V ≤ 5 in this universe

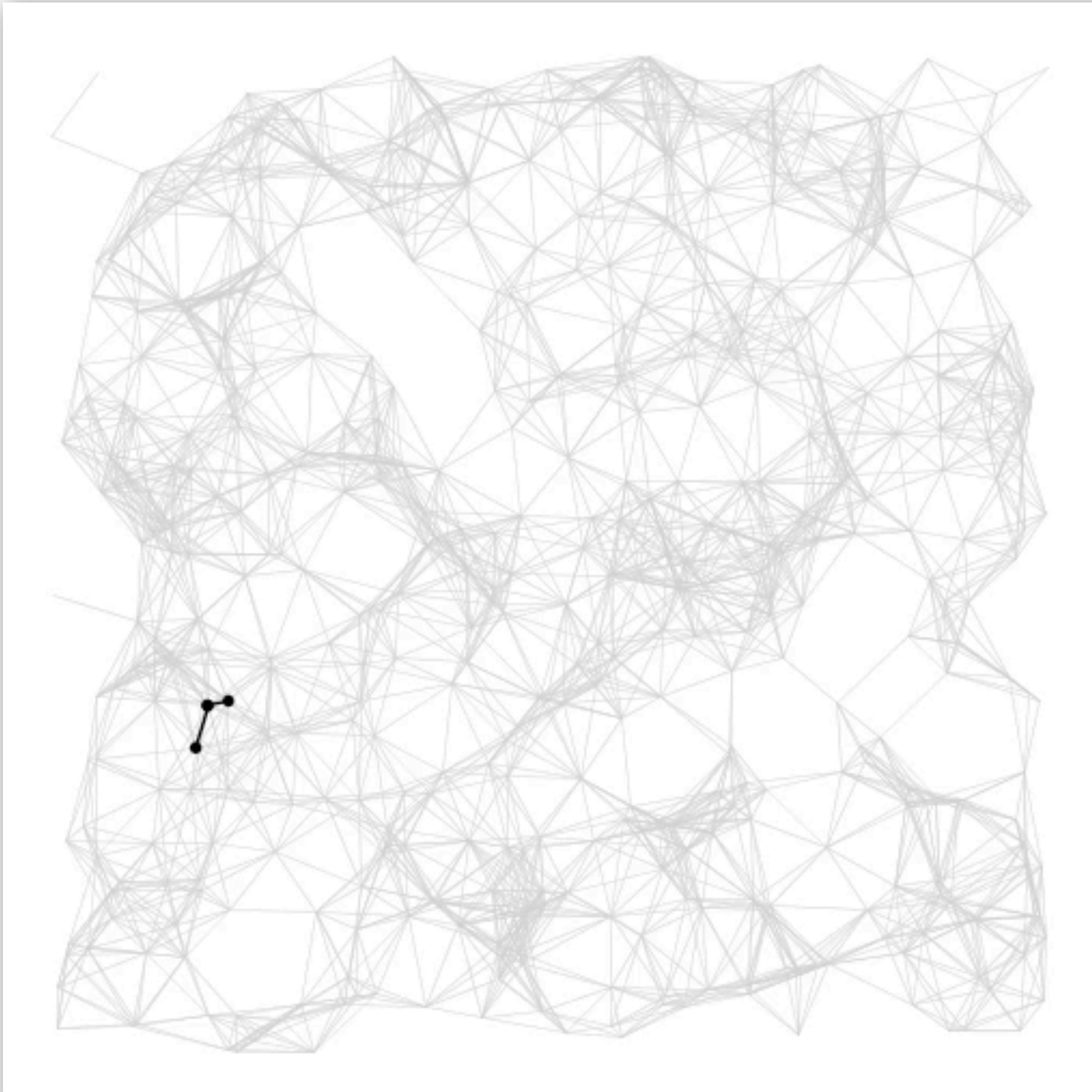
Remark. If edges are already sorted, order of growth is $E \log^* V$.

Prim's algorithm demo

Prim's algorithm. [Jarník 1930, Dijkstra 1957, Prim 1959]

- Start with vertex 0 and greedily grow tree T.
- At each step, add to T the min weight edge with exactly one endpoint in T.

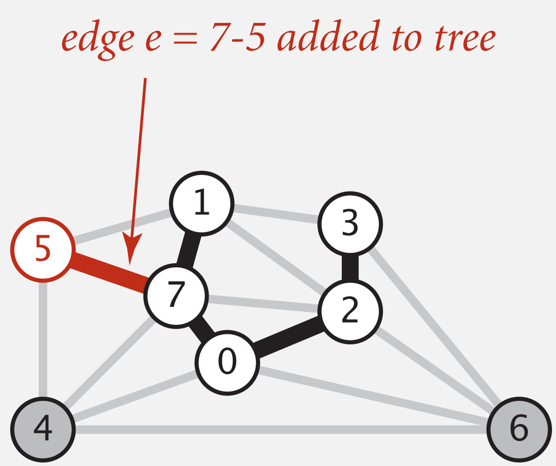
Prim's algorithm: visualization



Proposition. Prim's algorithm computes the MST.

Pf. Prim's algorithm is a special case of the greedy MST algorithm.

- Suppose edge e = min weight edge connecting a vertex on the tree to a vertex not on the tree.
- Cut = set of vertices connected on tree.
- No crossing edge is black.
- No crossing edge has lower weight.



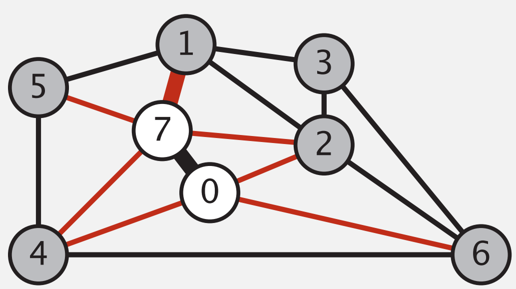
Prim's algorithm: implementation challenge

Challenge. Find the min weight edge with exactly one endpoint in T.

How difficult?

- E ← try all edges
- V
- $\log E$ ← use a priority queue !
- $\log^* E$
- 1

1-7 is min weight edge with exactly one endpoint in T



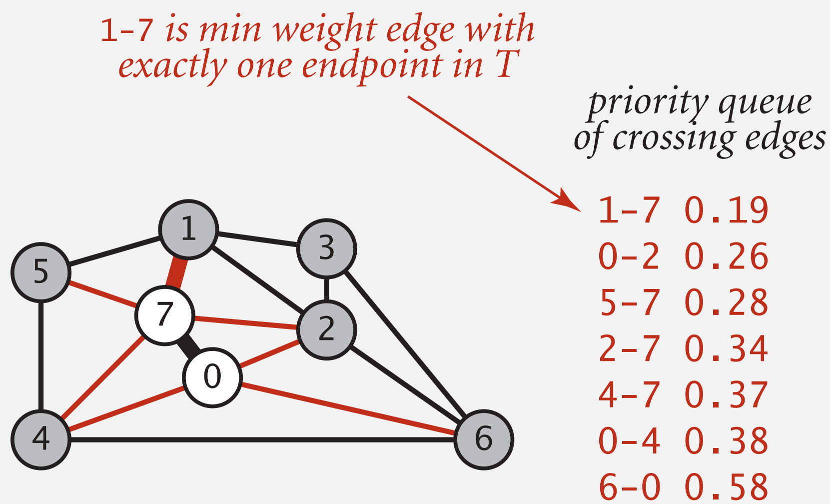

```
1-7 0.19
0-2 0.26
5-7 0.28
2-7 0.34
4-7 0.37
0-4 0.38
6-0 0.58
```

Prim's algorithm: lazy implementation

Challenge. Find the min weight edge with exactly one endpoint in T.

Lazy solution. Maintain a PQ of edges with (at least) one endpoint in T.

- Key = edge; priority = weight of edge.
- Delete-min to determine next edge $e = v–w$ to add to T.
- Disregard if both endpoints v and w are in T.
- Otherwise, let v be vertex not in T:
  - add to PQ any edge incident to v (assuming other endpoint not in T)
  - add v to T



Prim's algorithm demo: lazy implementation

Prim's algorithm: lazy implementation

```
public class LazyPrimMST
{
   private boolean[] marked;   // MST vertices
   private Queue<Edge> mst;    // MST edges
   private MinPQ<Edge> pq;     // PQ of edges

    public LazyPrimMST(WeightedGraph G)
    {
        pq = new MinPQ<Edge>();
        mst = new Queue<Edge>();
        marked = new boolean[G.V()];
        visit(G, 0);

        while (!pq.isEmpty())
        {
           Edge e = pq.delMin();
           int v = e.either(), w = e.other(v);
           if (marked[v] && marked[w]) continue;
           mst.enqueue(e);
           if (!marked[v]) visit(G, v);
           if (!marked[w]) visit(G, w);
        }
   }
}
```

assume G is connected

repeatedly delete the min weight edge e = v–w from PQ

ignore if both endpoints in T

add edge e to tree

add v or w to tree

Prim's algorithm: lazy implementation

```
private void visit(WeightedGraph G, int v)
{
   marked[v] = true;
   for (Edge e : G.adj(v))
      if (!marked[e.other(v)])
         pq.insert(e);
}

public Iterable<Edge> mst()
{  return mst;  }
```

add v to T

for each edge e = v–w, add to PQ if w not already in T

Lazy Prim's algorithm: running time

Proposition. Lazy Prim's algorithm computes the MST in time proportional to $E \log E$ and extra space proportional to E (in the worst case).

Pf.

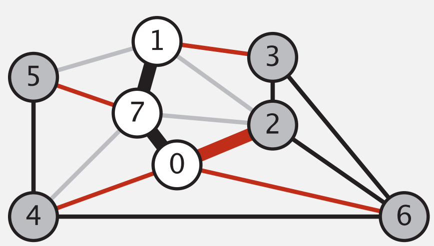
operation	frequency	binary heap
delete min	E	log E
insert	E	log E

Challenge. Find min weight edge with exactly one endpoint in T.

pq has at most one entry per vertex

Eager solution. Maintain a PQ of vertices connected by an edge to T, where priority of vertex v = weight of shortest edge connecting v to T.

- Delete min vertex v and add its associated edge $e = v\text{–}w$ to T.
- Update PQ by considering all edges $e = v\text{–}x$ incident to v
  - ignore if x is already in T
  - add x to PQ if not already on it
  - decrease priority of x if $v\text{–}x$ becomes shortest edge connecting x to T



0		
1	1-7	0.19
2	0-2	0.26
3	1-3	0.29
4	0-4	0.38
5	5-7	0.28
6	6-0	0.58
7	0-7	0.16

red: on PQ

black: on MST

Prim's algorithm: eager implementation demo

Use `IndexMinPQ`: key = edge weight, index = vertex.

(eager version has at most one PQ entry per vertex)

Indexed priority queue

Associate an index between 0 and $N - 1$ with each key in a priority queue.

- Client can insert and delete-the-minimum.
- Client can change the key by specifying the index.

`public class IndexMinPQ<Key extends Comparable<Key>>`

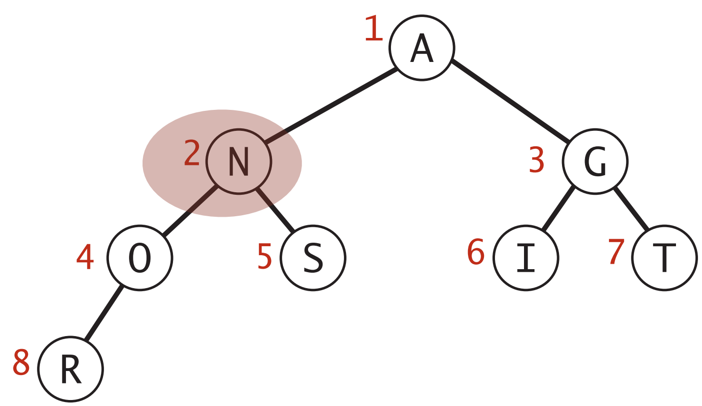
	`IndexMinPQ(int N)`	*create indexed priority queue with indices 0, 1, …, N-1*
`void`	`insert(int k, Key key)`	*associate key with index k*
`void`	`decreaseKey(int k, Key key)`	*decrease the key associated with index k*
`boolean`	`contains()`	*is k an index on the priority queue?*
`int`	`delMin()`	*remove a minimal key and return its associated index*
`boolean`	`isEmpty()`	*is the priority queue empty?*
`int`	`size()`	*number of entries in the priority queue*

Indexed priority queue implementation

Implementation.

- Start with same code as `MinPQ`.
- Maintain parallel arrays `keys[]`, `pq[]`, and `qp[]` so that:
  - `keys[i]` is the priority of `i`
  - `pq[i]` is the index of the key in heap position `i`
  - `qp[i]` is the heap position of the key with index `i`
- Use `swim(qp[k])` implement `decreaseKey(k, key)`.

i	0	1	2	3	4	5	6	7	8
keys[i]	A	S	O	R	T	I	N	G	-
pq[i]	-	0	6	7	2	1	5	4	3
qp[i]	1	5	4	8	7	6	2	3	-



Prim's algorithm: running time

Depends on PQ implementation: V insert, V delete-min, E decrease-key.

PQ implementation	insert	delete-min	decrease-key	total
array	1	V	1	V^2
binary heap	log V	log V	log V	E log V
d-way heap (Johnson 1975)	d $\log_d$ V	d $\log_d$ V	$\log_d$ V	E $\log_{E/V}$ V
Fibonacci heap (Fredman-Tarjan 1984)	1 †	log V †	1 †	E + V log V

† amortized

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

- edge-weighted graph API
- greedy algorithm
- Kruskal's algorithm
- Prim's algorithm
- **advanced topics**

Does a linear-time MST algorithm exist?

deterministic compare-based MST algorithms

year	worst case	discovered by
1975	E log log V	Yao
1976	E log log V	Cheriton-Tarjan
1984	E log* V, E + V log V	Fredman-Tarjan
1986	E log (log* V)	Gabow-Galil-Spencer-Tarjan
1997	E α(V) log α(V)	Chazelle
2000	E α(V)	Chazelle
2002	optimal	Pettie-Ramachandran
20xx	E	???

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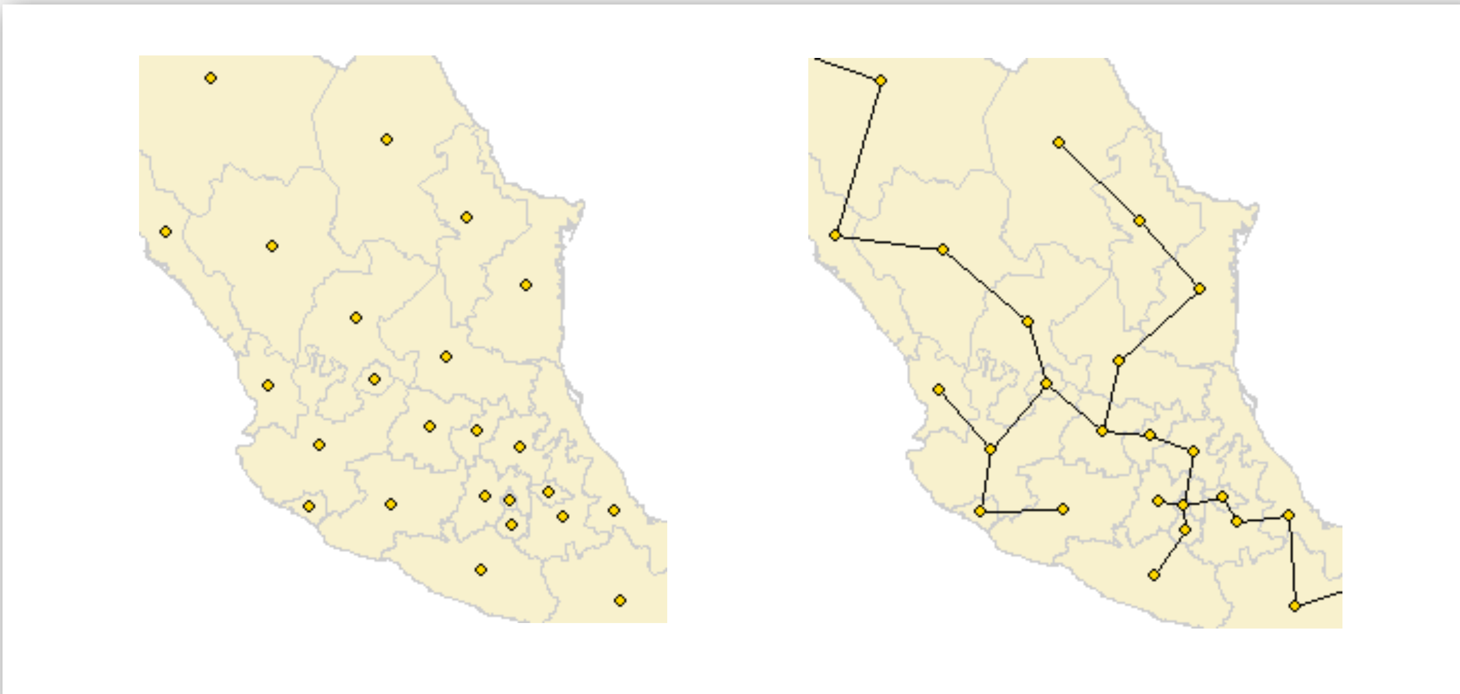
Remark. Linear-time randomized MST algorithm (Karger-Klein-Tarjan 1995).

Euclidean MST

Given N points in the plane, find MST connecting them, where the distances between point pairs are their Euclidean distances.



Brute force. Compute $\sim N^2 / 2$ distances and run Prim's algorithm.

Ingenuity. Exploit geometry and do it in $\sim c\, N \log N$.